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1. Introduction.

A great part of the theory of conformal mapping has been built around the study of the coefficients a_n of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots \tag{1}$$

schlicht in the interior $|z| < 1$ of the unit circle and the coefficients b_n of functions

$$g(z) = z + b_o + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots \tag{2}$$

schlicht in the exterior $|z| > 1$ of the unit circle. The estimates of these coefficients which have been obtained fall essentially into two classes, namely, those which follow in a natural way from the area theorem

$$1 \geq |b_1|^2 + 2|b_2|^2 + 3|b_3|^2 + \dots \tag{3}$$

or its variants, and those, such as Loewner's theorem $|a_3| \leq 3$, which cannot be derived from such elementary considerations. While most of the useful distortion theorems of conformal mapping are consequences of the area theorem (3), there is nevertheless a great interest attached to the more remote class of inequalities because of the unanswered status of the Bieberbach conjecture $|a_n| \leq n$.

The principal result of the present paper is the sharp inequality

$$(4) \qquad |b_3| \le \frac{1}{2} + e^{-6} \quad ,$$

which belongs in this latter category. For earlier coefficients, the estimates $|b_1| \le 1$ and $|a_2| \le 2$ follow quickly from (3), while even the more difficult inequality $|b_2| \le 2/3$ can be deduced from a generalized area principle. Next in order of difficulty comes the theorem $|a_3| \le 3$ due to Loewner [3], for which we shall give here a new and particularly simple proof. Thus the bound (4) on $|b_3|$ represents possibly the farthest point yet reached in estimating the higher coefficients of schlicht functions.

Our method of proving (4) is based on the differential equation for the schlicht function maximizing $|b_3|$ which results from an application of interior variations. We suppose that by now the precise derivation of such differential equations is familiar to the student of schlicht functions. Our contribution lies rather in determining the correct values of the parameters which appear in the differential equation, and this permits us to integrate the equation in closed form and find the largest value of $|b_3|$. Underlying our manipulations are a set of identities involving elliptic integrals which determine the parameters in the differential equation in such a way that its solution is a schlicht function. The main difficulties of the investigation center about a successful analysis of these identities. It is because the corresponding identities for the case of higher coefficients involve hyperelliptic integrals that the Bieberbach conjecture $|a_n| \le n$ remains an unsettled problem.

A special significance attaches to the sharp estimate (4) because this result forces rejection of the earlier conjecture [9] that

$$(5) \qquad |b_n| \le \frac{2}{n+1} \quad , \qquad n = 1, 2, 3, \ldots ,$$

with equality holding for essentially only the function

$$g(z) = (z^{n+1} + 2 + z^{-n-1})^{\frac{1}{n+1}} \quad . \tag{6}$$

While this mapping function is a solution of the differential equation and the associated parameter relations, we succeed nevertheless in finding, for $n = 3$, another solution with a larger third coefficient. Proof of (4) consists merely in showing that this new solution and (6) are actually the only functions fulfilling the requirements upon an extremal mapping. The existence of superfluous solutions of the differential equations again illustrates the difficulties inherent in the coefficient problem for schlicht functions and indicates that a naive approach through conjectures based on familiar elementary maps is of no avail. We emphasize, however, that our advance here does not cast doubt on the Bieberbach conjecture, since we obtain an extremal function for (4) which has real coefficients and the Bieberbach conjecture has already been established for functions with real coefficients.

In the next section, we illustrate our fundamental technique by giving a new proof of Loewner's theorem, based on the differential equation for the extremal function. The sections following are devoted to the more tedious proof of the inequality (4). Closing portions of the paper take up corollaries of the principal theorem, such as the inequalities

$$\mathrm{Re}\left\{b_3 - 3ib_1\right\} \leq 3 \quad , \quad \mathrm{Re}\left\{b_2 + 2b_1\right\} \leq 2 \quad , \tag{7}$$

or indicate results which fit appropriately within the broader scope of our investigation.

2. <u>Proof that</u> $|a_3| \leq 3$.

Each schlicht function $f(z)$ of the form (1) generates a schlicht function $g(z)$ of the form (2) according to the rule

(8) $$g(z) = \frac{1}{f(1/z)} = z - a_2 + \frac{a_2^2 - a_3}{z} + \cdots .$$

If the behavior of $g(z)$ on the unit circle is sufficiently regular, we find by the residue theorem

(9) $$b_o = \frac{1}{2\pi i} \oint_{|z|=1} g(z) \frac{dz}{z} ,$$

or, using (8) and setting $z = e^{i\theta}$,

(10) $$-a_2 = b_o = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{f(z)} d\theta .$$

With $t = g(z)$, we can consider the image Γ in the t-plane of the unit circle $|z| = 1$ and we can interpret the measure

(11) $$d\mu = \frac{1}{2\pi} d\theta$$

in the usual electrostatic sense. Thus we think of $d\mu$ as the natural charge distribution on Γ. It is a non-negative distribution of total charge 1, and we therefore call the coefficient

(12) $$b_o = \int_\Gamma t \, d\mu$$

the conformal centroid, or centroid, of the set Γ. Formula (10) shows that the coefficient a_2 is related to the centroid of Γ by

(13) $$-a_2 = \int_\Gamma t \, d\mu ,$$

and thus it is clear that $-a_2$ is a point inside the convex hull of Γ.

With these preliminaries behind us, we proceed to the problem of maximizing $|a_3|$. We write

(14) $$w = f(z)$$

and we choose for $f(z)$, without loss of generality, the extremal function maximizing $|a_3|$ such that $a_3 > 0$. It follows from the method of interior variation that the extremal function (14) satisfies the ordinary differential equation [4, 5, 7, 8]

(15) $$\frac{dw^2}{w^2}\left(\frac{1}{w^2} + \frac{2a_2}{w}\right) = \frac{dz^2}{z^2}\left(\frac{1}{z^2} + \frac{2a_2}{z} + 2a_3 + 2\bar{a}_2 z + z^2\right) ,$$

where the parenthesis on the right is non-negative for $|z| = 1$. Thus if we put

(16) $$t = \frac{1}{w} = \frac{1}{f(z)} ,$$

we find that the image Γ in the t-plane of the unit circle $|z| = 1$ consists of analytic arcs satisfying the differential equation

(17) $$\operatorname{Re}\left\{\frac{(t+2a_2)^{1/2}}{t^{1/2}}\,dt\right\} = 0 .$$

The conformal transformation

(18) $$H = \int_0^t \frac{(t+2a_2)^{1/2}}{t^{1/2}}\,dt$$

performs a univalent map of either of the half-planes bounded by the line L through the points 0 and $-2a_2$ in the t-plane onto a polygonal region R of the H-plane bounded by a linear ray from the origin, a finite line segment joining this ray at the origin under an angle of 90° with respect to the region, and a second infinite linear ray separating from the other end of the segment at an angle of 270° with respect to the region. The expression (18) is, in fact, merely a Schwarz-Christoffel transformation of a rotated half-plane. Now the arcs Γ in the t-plane correspond to a segment of the

imaginary axis in the H-plane, according to the differential equation (17). Furthermore, if this segment, starting out from the origin, enters one of the above regions R, it must remain there, since R consists of the sum of two quadrants. Thus the curve Γ must either coincide with the line L between 0 and $-2a_2$, possibly forking at $-2a_2$, or else, if we overlook the origin, Γ must lie entirely interior to one of the half-planes bounded by L. We shall exclude this latter possibility.

Indeed, if Γ lies in the interior of one of the two half-planes bounded by L, then so does its centroid with respect to the natural charge distribution (11). But from the explicit calculation (13), the centroid lies on the line L, halfway between 0 and $-2a_2$. Thus Γ can lie in no such half-plane and must actually coincide with L between 0 and $-2a_2$, with a possible fork at the latter point.

To exclude the fork, we notice that such a configuration would entail two end-points of Γ corresponding to two double zeros of the right-hand side of the differential equation (15). Thus we would have

$$(19) \qquad \left(\frac{1}{z^2} + \frac{2a_2}{z} + 2a_3 + 2\bar{a}_2 z + z^2\right) = \left(\frac{1}{z} + a_2 + z\right)^2 \quad ,$$

whence

$$(20) \qquad 2a_3 = a_2^2 + 2 \quad .$$

Since $|a_2| \leq 2$, this leads to the conclusion

$$(21) \qquad |a_3| \leq \frac{|a_2|^2}{2} + 1 \leq 3 \quad .$$

For equality to hold in (21) we must require $|a_2| = 2$, and this is true essentially only for the Koebe slit mapping

$$(22) \qquad w = \frac{z}{(1-z)^2} \quad .$$

Notice that (20) follows even when Γ does not fork at $-2a_2$, since in that case the right-hand side of (15) must have a quadruple root there.

This completes our proof of Loewner's theorem. It is based on an appropriate use of the identity (13), obtained from the schlicht character of the mapping (14), and it exploits in an essential way a geometrical analysis of the behavior of solutions of the differential equation (17).

3. The Inequality for b_3.

We proceed to the proof of (4) in several stages. Since the differential equation for the extremal function $g(z)$ maximizing $|b_3|$ and normalized so that $b_3 > 0$ is less familiar than the analogous differential equations for functions schlicht in the interior of the unit circle, we sketch a derivation of this equation. We stress that the derivation presented here is heuristic, and we refer to the literature for an exact treatment [7, 10].

The extremal function

$$t = g(z) \tag{23}$$

maps the unit circle $|z| = 1$ onto a system of curves Γ. There is no loss of generality if we assume throughout that $b_o = 0$, since this can be achieved simply by a translation of Γ. Let Γ_ρ be a small arc of Γ of outer mapping radius ρ and let t_o be a point of Γ_ρ. Then there is a conformal mapping of the form

$$\zeta = t - t_o + C_o\rho + \frac{C_1\rho^2}{(t-t_o)} + \frac{C_2\rho^3}{(t-t_o)^2} + \cdots \tag{24}$$

taking the exterior of Γ_ρ into the exterior $|\zeta| > \rho$ of the circle of radius ρ in the ζ-plane. We introduce the special functions

$$g^* = \zeta + \frac{B_1\rho^2}{\zeta} \quad , \tag{25}$$

with

(26) $$|B_1| \leq 1 \quad ,$$

which are schlicht in $|\zeta| > \rho$. The coefficient B_1 can be chosen arbitrarily except for the condition (26). It is well known that for each n the coefficients C_n are bounded uniformly in ρ.

By composition of the mappings (23), (24), and (25), we construct for $|z| > 1$ the schlicht function

(27) $$g^*(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \ldots - t_o + C_o\rho + \frac{(C_1+B_1)\rho^2}{g(z)-t_o} + o(\rho^2)$$

$$= z - t_o + C_o\rho + \frac{b_1+(C_1+B_1)\rho^2}{z} + \frac{b_2+t_o(C_1+B_1)\rho^2}{z^2} + \frac{b_3+(t_o^2-b_1)(C_1+B_1)\rho^2}{z^3}$$

$$+ \ldots + o(\rho^2) \quad .$$

From the extremal property of g(z) it follows that

(28) $$|b_3+(t_o^2-b_1)(C_1+B_1)\rho^2 + o(\rho^2)| \leq |b_3| \quad .$$

We let $\rho \to 0$ and we note that $C_1 \to -e^{2i\varphi}$, where φ is the angle of inclination of the tangent to Γ at t_o. Since $b_3 > 0$, we derive from (28) in the limit as $\rho \to 0$ the inequality

(29) $$\mathrm{Re}\left\{(t_o^2-b_1)(B_1-e^{2i\varphi})\right\} \leq 0 \quad ,$$

where B_1 is any complex number satisfying (26). Because of the freedom in the choice of B_1, the variational condition (29) yields the relation

(30) $$(t_o^2-b_1)dt_o^2 \geq 0$$

for the differential element dt_o of Γ at the point t_o. This result is actually a differential equation for the system of arcs Γ.

We can derive from (30) a differential equation for the extremal function g(z). Consider the expression

$$z^2g'(z)^2[g(z)^2-b_1] = z^4-b_1z^2-2b_2z-4b_3+\dots \quad , \tag{31}$$

which is an analytic function of z in the exterior of the unit circle, except for the indicated pole at infinity. According to (30), this function must be real for $|z|=1$, and hence we can continue it analytically into the interior of the unit circle by the Schwarz reflection principle. The function has a pole at the origin determined by the expansion (31), and hence we are able to calculate it explicitly and obtain

$$z^2 \frac{dg^2}{dz^2}(g^2-b_1) = z^4-b_1z^2-2b_2z-4b_3-\frac{2\bar{b}_2}{z}-\frac{\bar{b}_1}{z^2}+\frac{1}{z^4} \quad . \tag{32}$$

This is the desired differential equation for the schlicht function g, and we note only that, according to (30), the right-hand side must be non-positive for $|z|=1$.

We turn to the rigorous integration of (32). There are three characteristically different cases to be considered. (i) The curves Γ contain both the square roots of b_1. In general, Γ will fork at these points and will have four end-points, each of which corresponds to a double root of the right-hand side of (32) on the unit circle $|z|=1$. However, Γ might not fork and might even terminate at a critical point, but (32) will still have four double roots whenever Γ contains the two square roots of b_1. (ii) The curves Γ contain precisely one of the square roots of b_1; Γ will have, in general, three end-points, so that the right-hand side of (32) has only three double roots on $|z|=1$ and the two remaining roots lie at inverse points inside and outside the unit circle. (iii) The set Γ does not contain either of the square roots of b_1 and hence consists of a simple arc without forks and with only two end-points, so that the right-hand side of (32) has two double roots and four simple roots. In this last case, the integration

of (32) involves elliptic integrals, whereas in the first two cases only elementary integrals are required.

In later sections of the paper we shall prove that cases (ii) and (iii) can be excluded. We study in this section only case (i) and we determine the actual extremal function which maximizes $|b_3|$.

In case (i), the right-hand side of (32) has four double roots and is a perfect square, whence

$$z^4 - b_1 z^2 - 2b_2 z - 4b_3 - \frac{2\bar{b}_2}{z} - \frac{\bar{b}_1}{z^2} + \frac{1}{z^4} = \left(z^2 - \frac{b_1}{2} - \frac{1}{z^2}\right)^2 \tag{33}$$

and

$$b_2 = 0 \quad , \quad b_3 = \frac{1}{2} - \frac{b_1^2}{16} \quad . \tag{34}$$

The coefficient b_1 must be pure imaginary, since both sides of (33) are non-positive. We can now take the square root of both sides of (32) and integrate to obtain

$$\frac{g(g^2-b_1)^{1/2}}{2} - \frac{b_1}{2}\log\frac{g+(g^2-b_1)^{1/2}}{b_1^{1/2}} = \frac{z^2}{2} + \frac{1}{2z^2} - \frac{b_1}{2}\log z + K \quad , \tag{35}$$

where K is a constant of integration.

In order to evaluate K, we expand (35) about the point at infinity in the z-plane, using (2) and remembering that we took $b_o = 0$. By noting that the constant terms on both sides of the equation must be the same, we find that

$$K = \frac{3}{4} b_1 - \frac{b_1}{4}\log\frac{4}{b_1} \quad . \tag{36}$$

Since Γ passes through the point $b_1^{1/2}$, there exists a value z_o of z on the unit circle $|z| = 1$ such that

$$g(z_o) = b_1^{1/2} \quad . \tag{37}$$

Substitution of this value of z into (35) yields the additional relation

(38) $$0 = \frac{z_o^2}{2} + \frac{1}{2z_o^2} - \frac{b_1}{2} \log z_o + K \quad .$$

The terms in (38) involving z_o are real, and hence K is real. Therefore by (36) we have, since b_1 is pure imaginary,

(39) $$3b_1 + b_1 \log \frac{|b_1|}{4} = 0 \quad .$$

Using again the imaginary character of b_1, we deduce from equation (39) that either $b_1 = 0$ or else $|b_1| = 4e^{-3}$ and

(40) $$b_1 = 4ie^{-3} \quad .$$

The root $b_1 = 0$ of (39) leads to the solution (6) of (32), with $n = 3$, and $b_3 = 1/2$. On the other hand, the value (40) for b_1 leads by (34) to the value

(41) $$b_3 = \frac{1}{2} + e^{-6}$$

of the third coefficient of $g(z)$. This value is the larger of the two, and thus the function maximizing $|b_3|$ must be the one defined, according to (35), by the implicit relation

(42) $$g(g^2-4ie^{-3})^{1/2} - 4ie^{-3} \log \frac{g+(g^2-4ie^{-3})^{1/2}}{2z} = z^2 + \frac{1}{z^2} + 6ie^{-3} \quad .$$

The extremal function (42) maps the exterior of the unit circle in the z-plane onto the exterior of a system of arcs Γ which consists of a line segment joining the two square roots of $4ie^{-3}$ and four analytic arcs forking from these square roots at angles of 120^o. We remark that an extremal function with real coefficients can be obtained from the present one by rotation, and, indeed, $e^{(\pi i)/4} g(ze^{-(\pi i)/4})$ is such a function. However, its third coefficient is negative.

In order to establish that (41) is actually the largest value of b_3, we must exclude the above cases (ii) and (iii). This will be done in the next

sections by a method based on the knowledge that for schlicht solutions of (32), the singular points of the differential equation in the z-plane must correspond to the singular points in the t-plane.

4. Exclusion of Elliptic Integrals.

We establish in this section that there are no schlicht solutions of (32) in the case (iii) where the hypothesis is that the image Γ in the t-plane of the unit circle $|z|=1$ consists of a single analytic arc and does not pass through the branch points of (30) at the square roots of b_1. Actual integration of the right-hand side of (32) in this case would involve the use of elliptic integrals, and the success of our treatment stems from the fact that we are able to avoid such a step and work only with the left-hand side of the equation, or, more precisely, with (30). Our first remark is that we do not lose any generality if we suppose that the z-plane and the t-plane have been rotated so that $b_1 \geq 0$, while b_3 is no longer necessarily real. This new normalization is more convenient for our study of (30), but we notice that (30) must now be replaced by the more general differential equation

$$e^{2i\alpha}(t^2-b_1)dt^2 > 0 \tag{43}$$

for the arc Γ, where α is a fixed real parameter depending on the angle of rotation. The normalization $b_0 = 0$, made previously, is not altered by the rotation of coordinates.

We introduce the integral

$$H = \int_0^t (t^2-b_1)^{1/2}\, dt \tag{44}$$

and we point out that the differential equation (43) merely states that Γ is the image by the transformation (44) of a line segment L in the H-plane

inclined at an angle $-\alpha$ with the real axis. This interpretation of (43) permits us to show that Γ cuts the real axis between $-b_1^{1/2}$ and $+b_1^{1/2}$. In fact (44), viewed as a Schwarz-Christoffel transformation, maps the upper half of the t-plane onto the exterior of a semi-infinite strip of the form

$$\text{Re } H > 0 \quad , \quad -h < \text{Im } H < h \quad , \tag{45}$$

where

$$h = \text{Im} \int_0^{b_1^{1/2}} (t^2 - b_1)^{1/2}\, dt \quad . \tag{46}$$

In order to establish that Γ cuts the real axis between $-b_1^{1/2}$ and $+b_1^{1/2}$, it suffices to prove that Γ cuts both the real axis and the imaginary axis. For if this is true, then the line segment L in the H-plane corresponding to Γ must cut both the boundary of the semi-infinite strip (45) and the negative real axis in the H-plane, since these map by (44) into the real and imaginary axes in the t-plane. Any line segment L with the above properties has to intersect the segment Re H = 0, -h < Im H < h, and since this segment corresponds to the interval between $-b_1^{1/2}$ and $+b_1^{1/2}$, the arc Γ must cut that interval.

It remains, then, to show that Γ cuts both the real axis and the imaginary axis. But, in the notation of formulas (9) and (12), we interpret the normalization $b_o = 0$ to mean that the centroid b_o of Γ lies at the origin, or in other words

$$b_o = \int_\Gamma t\, d\mu = 0 \quad . \tag{47}$$

This can only occur if Γ cuts both coordinate axes, and thus our lemma is established. A special application of this argument shows in addition that $b_1 \neq 0$.

We must consider all positions of the arc Γ which are consistent with the properties that it cuts the real axis between $-b_1^{1/2}$ and $+b_1^{1/2}$, that it

has outer mapping radius 1, and that its centroid lies at the origin. We shall prove that these properties imply that Γ passes through the origin and is symmetric in the origin. It is evident that the converse is true, namely, that an arc which satisfies the differential equation (43) and is symmetric in the origin must have its centroid at the origin. Thus our proof can be carried out by establishing the uniqueness for each value of the parameter α of this solution of the equation (47).

As a preliminary, we derive variational formulas for the capacity γ and centroid b_o of the curve Γ. We shall need such formulas for a shift of Γ corresponding to infinitesimal translation and magnification of the image segment L in the H-plane. We denote the shifted segment by L^* and we let Γ^* denote the correspondingly varied arc in the t-plane. We shall use addition of an asterisk to indicate all quantities associated with the varied configuration. The infinitesimal transformation carrying L into L^* can be written in the form

$$H^* = (1+\epsilon_1)H+\epsilon_2 \quad , \tag{48}$$

where ϵ_1 is a small real number and ϵ_2 is a small complex number. We set $\epsilon = \max\,(|\epsilon_1|, |\epsilon_2|)$.

We denote by p(t) the analytic function whose real part is the Green's function of the exterior of Γ with pole at infinity. Thus near infinity

$$\mathrm{Re}\ p(t) = \log|t| - \gamma + O(\frac{1}{|t|}) \quad , \tag{49}$$

and e^{γ} is the outer mapping radius of Γ. We let $\varphi(t)$ be the function, analytic in the exterior of Γ, which has a pole of the form

$$\varphi(t) = t + c_o + \frac{c_1}{t} + \frac{c_2}{t^2} + \cdots \tag{50}$$

at infinity and which has real boundary values on Γ. Similarly, $\psi(t)$ is defined to be the function, analytic in the exterior of Γ, which has a pole

of the form

(51) $$\psi(t) = t + d_o + \frac{d_1}{t} + \frac{d_2}{t^2} + \cdots$$

at infinity and which has imaginary boundary values on Γ. The centroid b_o of Γ can be expressed in terms of the expansions (50) and (51) by the formula

(52) $$b_o = -\mathrm{Re}\, d_o - i\, \mathrm{Im}\, c_o = \beta_1 + i\beta_2 \quad ,$$

since, in terms of the mapping (23),

(53) $$\varphi = z + \frac{1}{z} + \text{real const.} \quad ,$$

(54) $$\psi = z - \frac{1}{z} + \text{imaginary const.} \quad .$$

Variational formulas for the capacity γ and centroid b_o can be found using the domain functions p, φ, and ψ. We denote by $t^*(t)$ the infinitesimal transformation of Γ onto Γ^* induced by (44) and (48). By the residue theorem, we find

(55) $$\gamma^* - \gamma = \mathrm{Re}\, \frac{1}{2\pi i} \oint [p(t) - p^*(t)]dp(t) \quad ,$$

where the path of integration is a closed curve surrounding Γ. Since p(t) and $p^*(t^*(t))$ are pure imaginary on Γ, we can rewrite (55) in the form

(56) $$\gamma^* - \gamma = \mathrm{Re}\, \frac{1}{2\pi i} \oint [p^*(t^*) - p^*(t)]dp(t) \quad .$$

We wish to evaluate the integral over the corresponding path in the H-plane, and thus we replace t as the independent variable by H to obtain

(57) $$\gamma^* - \gamma = \mathrm{Re}\, \frac{1}{2\pi i} \oint [p^*(H^*) - p^*(H)]dp(H) \quad ,$$

where we have used a loose notation that does not indicate explicitly the change in the functional form of p under the transformation from the t-plane to the H-plane.

From (48) and (57) we derive the variational formula

(58) $$\gamma^* - \gamma = \mathrm{Re}\, \frac{1}{2\pi i} \oint (\epsilon_1 H + \epsilon_2) p'(H)^2 dH + o(\epsilon) \quad ,$$

where the path of integration is a curve in the H-plane enclosing the line segment L. In a similar way, we derive from the expressions

(59) $$\beta_1^* - \beta_1 = \operatorname{Re} \frac{1}{2\pi i} \oint [\Psi(t) - \Psi^*(t)] dp(t) \quad ,$$

(60) $$\beta_2^* - \beta_2 = \operatorname{Im} \frac{1}{2\pi i} \oint [\varphi(t) - \varphi^*(t)] dp(t)$$

the variational formulas

(61) $$\beta_1^* - \beta_1 = \operatorname{Re} \frac{1}{2\pi i} \oint (\epsilon_1 H + \epsilon_2) \Psi'(H) p'(H) dH + o(\epsilon) \quad ,$$

(62) $$\beta_2^* - \beta_2 = \operatorname{Im} \frac{1}{2\pi i} \oint (\epsilon_1 H + \epsilon_2) \varphi'(H) p'(H) dH + o(\epsilon)$$

for the real and imaginary parts β_1 and β_2 of the centroid b_o of Γ.

Formula (58) can be simplified by evaluating the integral in the t-plane and by deforming the contour of integration into a path which consists of two small circles Ω_1 and Ω_2 about the end-points t_1 and t_2 of Γ and of the two edges of Γ joining these circles. In order to calculate the integrals over Ω_1 and Ω_2, we introduce the local uniformizers $\omega_1 = 2(t-t_1)^{1/2}$ and $\omega_2 = 2(t-t_2)^{1/2}$. We let the radii of the circles Ω_1 and Ω_2 tend to zero. The limit of the integral over Ω_k is $\operatorname{Re}\left\{ \delta t_k \left(\frac{dp}{d\omega_k}\right)^2 \right\}$, $k = 1,2$, where δt_k is the displacement of t_k under the shift (48). The integral over the two edges of Γ has the limiting value $(2\pi)^{-1} \int_\Gamma (\partial p/\partial \nu)^2 \delta\nu \, ds$, where ν is the inner normal and s is the arc length along Γ, where the integration is carried out along both edges of Γ, and where $\delta\nu$ is the normal displacement of Γ under the shift (48). We denote by $\partial p/\partial \nu_k$ the normal derivative of the function p in the plane of the uniformizer ω_k and we denote by $\delta\nu_k$ the tangential projection with respect to Γ of the shift δt_k, $k = 1,2$. Thus we obtain from (58) the Hadamard formula

(63) $$\delta\gamma = \frac{1}{2\pi} \int_\Gamma \left(\frac{\partial p}{\partial \nu}\right)^2 \delta\nu \, ds + \left(\frac{\partial p}{\partial \nu_1}\right)^2 \delta\nu_1 + \left(\frac{\partial p}{\partial \nu_2}\right)^2 \delta\nu_2 + o(\epsilon) \quad ,$$

and similarly, in an analogous notation, (61) and (62) yield

(64) $$\delta\beta_1 = \frac{1}{2\pi}\int_\Gamma \frac{\partial p}{\partial \nu}\frac{\partial \psi}{\partial \nu}\,\delta\nu\,ds + \frac{\partial p}{\partial \nu_1}\frac{\partial \psi}{\partial \nu_1}\,\delta\nu_1 + \frac{\partial p}{\partial \nu_2}\frac{\partial \psi}{\partial \nu_2}\,\delta\nu_2 + o(\epsilon)\ ,$$

(65) $$\delta\beta_2 = \frac{1}{2\pi i}\int_\Gamma \frac{\partial p}{\partial \nu}\frac{\partial \varphi}{\partial \nu}\,\delta\nu\,ds + \frac{1}{i}\frac{\partial p}{\partial \nu_1}\frac{\partial \varphi}{\partial \nu_1}\,\delta\nu_1 + \frac{1}{i}\frac{\partial p}{\partial \nu_2}\frac{\partial \varphi}{\partial \nu_2}\,\delta\nu_2 + o(\epsilon)\ ,$$

where $\delta\gamma = \gamma^* - \gamma$, $\delta\beta_1 = \beta_1^* - \beta_1$, and $\delta\beta_2 = \beta_2^* - \beta_2$. Notice that the derivatives $\partial p/\partial\nu$, $\partial\psi/\partial\nu$ and $-i\,\partial\varphi/\partial\nu$ are actually real numbers.

We attempt to choose the arc Γ as a solution of the differential equation (43) which cuts between $-b_1^{1/2}$ and $+b_1^{1/2}$ in such a way that the three equations

(66) $$\gamma = 0\ ,\qquad \beta_1 = 0\ ,\qquad \beta_2 = 0$$

are fulfilled. Given the upper end-point t_1 of Γ, the equation $\gamma = 0$ clearly determines the lower end-point t_2, since γ is a monotonic domain functional. We now establish that, along a prescribed curve solving (43), the equation $\beta_2 = 0$ determines the upper end-point t_1 uniquely. This is a consequence of the variational formula (65) because, as we shall prove in a moment,

(67) $$\frac{1}{i}\frac{\partial \varphi}{\partial \nu_1} > 0\ ,\qquad \frac{1}{i}\frac{\partial \varphi}{\partial \nu_2} < 0\ .$$

The inequalities (67) show that as t_1 rises along a solution of (43), and as t_2 follows t_1 so that $\gamma = 0$, the quantity β_2 increases monotonically. Thus, indeed, β_2 vanishes only once and t_1 is uniquely determined.

It remains to establish the inequalities (67). We prove first that our solution of (43) is a curve which intersects each horizontal line just once. Indeed, if such a curve intersects a horizontal line ℓ more than once,

it intersects it essentially at least three times, and between three intersections on the horizontal line ℓ there will be two points where $\operatorname{Im}\left\{e^{i\alpha}H\right\}$ is stationary, by Rolle's theorem. There is on the same horizontal line ℓ an additional stationary value of $\operatorname{Im}\left\{e^{i\alpha}H\right\}$ which occurs between the two intersections with the line ℓ of one of the solutions of (43) which forks through $-b_1^{1/2}$ or $+b_1^{1/2}$. This accounts for at least three stationary values of $\operatorname{Im}\left\{e^{i\alpha}H\right\}$ on the single horizontal line ℓ, and at each such stationary point we find from differentiation of (44)

$$\operatorname{Im} e^{2i\alpha}(t^2-b_1) = 0 \quad . \tag{68}$$

For t on the line ℓ, (68) reduces to a real quadratic equation, and thus it can have at most two roots. Thus three stationary values could not appear, and we have proved that a solution of (43) which cuts the real axis between $-b_1^{1/2}$ and $+b_1^{1/2}$ must cut each horizontal line just once.

The inequalities (67) can now be deduced from the maximum principle. On the slit Γ the function $\operatorname{Im}\left\{\varphi(t)-t+t_1\right\}$ is non-negative because $\operatorname{Im}\left\{\varphi\right\}=0$ and $\operatorname{Im}\left\{t-t_1\right\}\leq 0$ there. Hence, by the maximum principle, this function is positive in the exterior of Γ, and the first inequality (67) follows by differentiation when we note that at $t=t_1$ we have $\operatorname{Im}\left\{\varphi(t)-t+t_1\right\}=0$. Similarly, the harmonic function $\operatorname{Im}\left\{\varphi(t)-t+t_2\right\}$ is negative in the exterior of Γ because it is non-positive on Γ, and since this function vanishes at $t=t_2$, we obtain the second inequality (67).

Thus we have shown that the equations $\gamma=0$, $\beta_2=0$ determine a unique arc Γ on each level curve

$$\operatorname{Im}\left\{e^{i\alpha}H(t)\right\} = \lambda \tag{69}$$

cutting between $-b_1^{1/2}$ and $+b_1^{1/2}$. The problem is therefore to find all the values of λ such that $\beta_1=0$. It is clear from (44) that when $\lambda=0$ the

arc Γ satisfying $\gamma = \beta_2 = 0$ passes through the origin and is symmetric in the origin, so that $\beta_1 = 0$. We shall prove that this is the only choice for λ which gives $\beta_1 = 0$. Suppose, indeed, that there were another value λ_o of λ for which we could find an arc of the level curve (69) satisfying all the equations (66). Then we could vary λ between 0 and λ_o and consider for each intermediate value of λ the arc (69) with $\gamma = \beta_2 = 0$. By Rolle's theorem, there would exist an intermediate value λ_1 of λ for which β_1 would be stationary. Thus for the corresponding arc Γ and an appropriate variation of the type (48) we would obtain

(70) $$\delta\gamma = \delta\beta_1 = \delta\beta_2 = 0 \quad .$$

We shall establish that this is impossible, and therefore that the solution of (66) is unique.

For arbitrary real values of X, Y, and Z we consider the expression

(71) $$q(t) = Xp(t) + Y\psi(t) + Z\frac{\varphi(t)}{i} \quad .$$

We can choose X, Y, and Z so that at the end-points t_1 and t_2 of the arc Γ satisfying (70) the conditions

(72) $$\frac{\partial q}{\partial \nu_1} = 0 \quad , \quad \frac{\partial q}{\partial \nu_2} = 0$$

are fulfilled. This follows because $\partial q/\partial \nu$ is proportional to a linear combination of the functions 1, cos θ, and sin θ on the unit circle $z = e^{i\theta}$ in the z-plane. By the argument principle, (72) accounts for all the zeros of the real quantity $\partial q/\partial \nu$.

With this choice of X, Y, and Z, the variational formulas (63), (64), and (65) yield for any shift (48) of Γ the simple relation

(73) $$X\delta\gamma + Y\delta\beta_1 + Z\delta\beta_2 = \frac{1}{2\pi}\int_\Gamma \frac{\partial p}{\partial \nu}\frac{\partial q}{\partial \nu}\,\delta\nu\, ds + o(\epsilon) \quad .$$

However, the right-hand side of (73) does not vanish for the shift $\delta\nu$ which gave (70), because $\delta\nu$ is of one sign on one side of Γ and of the opposite sign on the other side of Γ for such a shift, and because by (72) the same is true of $\partial q/\partial\nu$. Thus we arrive at a contradiction.

Therefore, we conclude that the curve Γ must be symmetric in the origin. Next, if the parameter α of formula (69) lies in the interval $0 < \alpha < \pi/2$, we arrive at a contradiction because of the identity

$$2b_1 = \frac{1}{2\pi i} \int_{|z|=1} g(z)^2 \frac{dz}{z} = \int_{\Gamma} t^2 d\mu \qquad (74)$$

Indeed, the left-hand side of (74) is real and positive, whereas the right-hand side must have a positive imaginary part because Γ lies in the first and third quadrants. A similar conclusion holds when $-\frac{\pi}{2} < \alpha < 0$, and when $\alpha = 0$ a contradiction is obtained because the two sides of (74) are real with opposite signs. Finally, the case $\alpha = \pi/2$ must be excluded also, because now in (74) we have $t^2 < b_1$ and the right-hand side is actually smaller than $2b_1$.

We thus obtain the final result that a single arc Γ without forks cannot occur in the solution of (32). We have therefore established that the case (iii) does not appear, and it remains only to exclude the case (ii) in order to prove the original inquality (4).

5. <u>The Equations for</u> b_1 <u>and</u> b_2.

Our treatment of case (ii) is based on direct integration of the differential equation (32). In the differential equation for any coefficient inequality, one can always find the correct number of conditions to determine all the coefficients which appear by expressing the fact that the singularities in the z-plane and the t-plane have to match up. In general, this procedure

is not feasible because it involves hyperelliptic integrals, and even in the case (iii) it would have led to elliptic integrals. However, we are able to succeed with the method in case (ii), since the integrations can be executed in terms of elementary integrals.

The hypothesis in case (ii) is that the boundary Γ in the t-plane consists of three analytic arcs forking from one of the branch points $-b_1^{1/2}$ or $+b_1^{1/2}$ at angles of 120°. Since such a system of arcs has three end-points, the right-hand side of (32) must have three double zeros and it can be represented in the form

$$(75)\quad z^4-b_1z^2-2b_2z-4b_3-\frac{2\bar{b}_2}{z}-\frac{\bar{b}_1}{z^2}+\frac{1}{z^4}=(E_1z+E_2+\frac{E_3}{z}+\frac{E_4}{z^2})^2(z^2-2rkz+k^2)\ .$$

Since the roots of the left-hand side of (75) lie at inverse points in the unit circle, we must have

$$(76)\qquad |k| = 1\quad ,\quad r > 1\ .$$

Furthermore, the coefficients of z^4, z^3, z^{-3}, and z^{-4} on the left in (75) are known, so we obtain four conditions giving the coefficients E_1, E_2, E_3, and E_4 in terms of the two parameters r and k. Thus

$$(77)\quad z^4-b_1z^2-2b_2z-4b_3-\frac{2\bar{b}_2}{z}-\frac{\bar{b}_1}{z^2}+\frac{1}{z^4}=(z+rk+\frac{r\bar{k}^2}{z}+\frac{\bar{k}}{z^2})^2(z^2-2rkz+k^2)\ .$$

By (77), we can express the three coefficients b_1, b_2, and b_3 in terms of the two real parameters r and arg k. In particular,

$$(78)\qquad b_1 = 3r^2k^2-k^2-2r\bar{k}^2\quad ,$$

and b_2 and b_3 can be expressed in terms of b_1.

In order to determine the unknown b_1, we have to integrate (32) explicitly. By (77), we can write (32) in the form

$$(79)\qquad (g^2-b_1)^{1/2}dg = (1+\frac{rk}{z}+\frac{r\bar{k}^2}{z^2}+\frac{\bar{k}}{z^3})(z^2-2rkz+k^2)^{1/2}dz\quad .$$

An equation for b_1 will result essentially from the fact that in the conformal transformation $t = g(z)$ one of the roots of $z^2-2rkz+k^2 = 0$ must map into a root of $t^2-b_1 = 0$. We introduce the notation

(80) $$W = (z^2-2rkz+k^2)^{1/2} \quad ,$$

and we integrate (79) to obtain

(81) $$\frac{g(g^2-b_1)^{1/2}}{2} - \frac{b_1}{2}\log[g+(g^2-b_1)^{1/2}]+K$$
$$= -\frac{b_1}{2}\log[W+z-rk]+\frac{\bar{b}_1}{2}\log[r-\frac{W+k}{z}]+\frac{W}{2}(z+rk-\frac{r\bar{k}^2}{z}-\frac{\bar{k}}{z^2}) \quad ,$$

where K is a constant of integration. The reader can easily check (81) by direct differentiation.

The best way to evaluate K is to substitute for z in (81) a root of the equation $W = 0$; the corresponding value of g is $b_1^{1/2}$. This yields

(82) $$K = \frac{b_1}{4}\log\frac{b_1}{k^2(r^2-1)} + \frac{\bar{b}_1}{4}\log(r^2-1) \quad .$$

On the other hand, K can also be evaluated by expanding both sides of (81) about the point at infinity. We have thus

(83) $$\frac{g^2}{2} - \frac{b_1}{4} - \frac{b_1}{2}\log 2g + K + 0(\frac{1}{|g|})$$
$$= \frac{z^2}{2} - \frac{3r^2k^2-k^2+2r\bar{k}^2}{4} + \frac{\bar{b}_1}{2}\log(r-1) - \frac{b_1}{2}\log 2z + 0(\frac{1}{|z|}) \quad ,$$

or, substituting for g the expansion (2) and letting $z \to \infty$,

(84) $$K = -\frac{3b_1+3r^2k^2-k^2+2r\bar{k}^2}{4} + \frac{\bar{b}_1}{2}\log(r-1)$$
$$= -b_1-r\bar{k}^2 + \frac{\bar{b}_1}{2}\log(r-1) \quad .$$

The equations (78), (82), and (84) yield together a single equation for the determination of b_1. We prefer, however, to eliminate K and b_1 in order

to obtain the one complex equation

$$(85)\qquad \frac{k^2+\bar{k}^2}{4}(3r+1)(r-1)\log(r-1)+\frac{k^2-\bar{k}^2}{4}(3r-1)(r+1)\log(r+1)$$

$$=\frac{(3r^2-1)k^2-2r\bar{k}^2}{4}\log[3r^2-1-2r\bar{k}^4]+(3r^2-1)k^2-r\bar{k}^2$$

for the two real unknowns r and $\arg k$. Our objective is to prove that (85) has no solutions consistent with the hypothesis (76).

It is necessary to specify which branches of the logarithms are meant in equation (85). This question can be discussed by a more careful examination of our derivation of (85). We first perform rotations through 90° and reflections in the z-plane and the t-plane until b_1 lies in the first quadrant, while b_3 remains positive. Taking (82) into account, we then rewrite (81) in the form

$$(86)\qquad g(g^2-b_1)^{1/2}-b_1\log\frac{[g+(g^2-b_1)^{1/2}](r^2-1)^{1/2}k}{[W+z-rk]b_1^{1/2}}$$

$$=\bar{b}_1\log\frac{rz-W-k}{z(r^2-1)^{1/2}}+W(z+rk-\frac{r\bar{k}^2}{z}-\frac{\bar{k}}{z^2}) \quad ,$$

where it is now correct to take values of all the logarithms so that they vanish at the root z of the equations $W=0$, $g=b_1^{1/2}$. In order to obtain (85), we let z increase from this root and become infinite along the ray $\arg z=\arg k$. A difficulty is encountered when we try to locate the corresponding trajectory of g.

The path covered by g obviously has the equation

$$(87)\qquad \operatorname{Im} p(g) = \arg k \quad .$$

Also, we have the relation

$$(88)\qquad p'(g) = \int_\Gamma \frac{d\mu}{g-t} \quad ,$$

so that the trajectory (87) is always directed away from the convex hull of Γ. Therefore, this trajectory cannot cross the ray $\arg g = \arg b_1^{1/2} + \pi$ beyond the point $-b_1^{1/2}$, since it starts out at the point $g = b_1^{1/2}$ and such a crossing would require an intermediate position in which its tangent does not cut Γ. Our analysis shows, then, that we are permitted to use only values of the logarithms in (86) which have an imaginary part lying between $-\pi$ and $+\pi$. It follows that in the final identity (85) the values of the logarithms should be chosen from this same principal branch. In particular, the logarithms on the left in (85) are real numbers.

Having determined the correct branch of the logarithms in (85), we proceed to pare down the region of variation of the parameters r and k which must be considered. We note that in addition to (78), the formula (77) yields the expressions

$$b_2 = (r^3-r)k^3+(r^2-1)\bar{k} \quad , \tag{89}$$

$$b_3 = r^3-r^2\,\frac{k^4+\bar{k}^4}{4} \tag{90}$$

for the coefficients b_2 and b_3. Substituting these results into the area theorem (3), we find

$$\left|3r^2-1-2r\bar{k}^4\right|^2+2\left|r^3-r+(r^2-1)\bar{k}^4\right|^2+3\left|r^3-r^2\,\frac{k^4+\bar{k}^4}{4}\right|^2 \le 1 \quad . \tag{91}$$

From (76) and (91) we find easily that $\bar{k}^4$ lies in the right half-plane and that $4r^2[\operatorname{Im}\bar{k}^4]^2+3r^4/4 \le 1$, whence

$$\left|\operatorname{Im}\bar{k}^4\right| < \frac{1}{4} \quad . \tag{92}$$

Thus we can replace (91) by the weaker estimate

$$(3r^2-2r-1)^2+2r^2(r^2-1)^2+3r^4/4 \le 1 \quad ,$$

from which follows

$$(3r+1)^2(r-1)^2+2(r+1)^2(r-1)^2 \le 1-3r^4/4 \le 1/4 \quad ,$$

or, finally,

(93) $$(r-1)^2[(3r+1)^2+8] \le \frac{1}{4} \quad .$$

We write $r-1=\epsilon$ and derive from (93) the inequality

(94) $$\epsilon^2+\epsilon^3 \le \frac{1}{96} \quad ,$$

whence

(95) $$\epsilon < \frac{1}{10} \quad .$$

Rearranging the terms in (85) in a more suggestive notation, we must prove that the equation

(96) $$(1+\bar{k}^4)(4+3\epsilon)\epsilon\log\epsilon+(1-\bar{k}^4)(2+3\epsilon)(2+\epsilon)\log(2+\epsilon)$$
$$= b_1\bar{k}^2\log b_1\bar{k}^2+8+24\epsilon+12\epsilon^2-(4+4\epsilon)\bar{k}^4$$

has no solutions in the region defined by (76), (92), and (95), when we use the principal branch of the logarithm. From (92) and (95) we obtain readily

(97) $$\mathrm{Re}\left\{(1+\bar{k}^4)(4+3\epsilon)\epsilon\log\epsilon\right\} < 0 \quad ,$$

(98) $$\mathrm{Re}\left\{(1-\bar{k}^4)(2+3\epsilon)(2+\epsilon)\log(2+\epsilon)\right\} < 1/2 \quad ,$$

and

(99) $$\left|\arg b_1\bar{k}^2\right| = \left|\arg[2+6\epsilon+3\epsilon^2-(2+2\epsilon)\bar{k}^4]\right| \le \pi/2 \quad .$$

By (99) and the estimate $|b_1| \le 1$, we have

(100) $$\left|b_1\bar{k}^2\log b_1\bar{k}^2\right| < \frac{1}{e}+\frac{\pi}{2} \quad ,$$

since

(101) $$\max_{0<x<1}\left|x\log x\right| = \frac{1}{e} \quad .$$

Finally, it is clear that

(102) $$\mathrm{Re}\left\{8+24\epsilon+12\epsilon^2-(4+4\epsilon)\bar{k}^4\right\} > \quad .$$

Thus, combining (97), (98), (100), and (102), we derive from (96) the absurd inequality

$$(103) \qquad 4 < \frac{1}{\epsilon} + \frac{\pi}{2} + \frac{1}{2} < 2.6 \quad ,$$

which shows that (85) has, indeed, no relevant solutions.

This finishes our proof that the case (ii) in the integration of the differential equation (32) does not actually arise, and thus, finally, we complete in every detail our proof of the original inequality (4). The tedious calculations required to bring us from the fundamental equation (85) determining b_1 to the contradiction (103) should not be allowed to obscure the basic value of the method.

6. <u>Corollaries of the Main Theorem</u>.

Because we know that (42) is the extremal function for the inequality (4), we can make variations in the large and derive a set of further inequalities by composition of suitable mappings. We carry out one example of this type here and establish the inequalities (7).

We write the extremal function (42) in the form

$$(104) \qquad t = z + \frac{4ie^{-3}}{z} + \frac{\frac{1}{2} + e^{-6}}{z^3} + \cdots \quad .$$

The boundary Γ in the t-plane includes for this mapping the segment from $-2\exp(-\frac{3}{2} + \frac{\pi i}{4})$ to $+2\exp(-\frac{3}{2} + \frac{\pi i}{4})$. The exterior of this segment is mapped onto the exterior of the circle of radius $\exp(-\frac{3}{2})$ in the w-plane by the transformation

$$(105) \qquad t = w + \frac{ie^{-3}}{w} \quad .$$

In the region $|w| > \exp(-\frac{3}{2})$, we can consider the schlicht function

$$(106) \qquad g^* = w + \frac{b_1 e^{-3}}{w} + \frac{b_2 e^{-\frac{9}{2}}}{w^2} + \frac{b_3 e^{-6}}{w^3} + \cdots \quad ,$$

where the numbers b_n are arbitrary coefficients of a schlicht function of the form (2). Through composition of the transformations (104), (105), and (106) we obtain outside the unit circle in the z-plane the schlicht function

$$(107) \qquad g^* = z + \frac{b_1 e^{-3} + 3ie^{-3}}{z} + \frac{b_3 e^{-6} - 3ib_1 e^{-6} - 2e^{-6} + \frac{1}{2}}{z^3} + \dots \quad .$$

Since g^* is a competing function for the inequality (4), the expansion (107) yields

$$(108) \qquad \mathrm{Re}\left\{ b_3 e^{-6} - 3ib_1 e^{-6} - 2e^{-6} + \frac{1}{2} \right\} \leq \frac{1}{2} + e^{-6} \quad ,$$

and the first inequality (7) is an immediate consequence of (108).

In a similar fashion we can derive the second inequality (7) from the knowledge that (6), with $n = 2$, is the extremal function for the inequality $|b_2| \leq 2/3$. The chief interest in the inequalities (7) stems from the fact that, in the sense of the substitution (8), they become equalities for the Koebe function.

The problem of maximizing b_3 when all the coefficients b_n of the schlicht function $g(z)$ are real has a somewhat unexpected solution, due to the fact that the extremal function (104) does not have real coefficients. In the case where the coefficients are real, routine application of variational methods leads to the same differential equation (32) which we obtained in the general case. One has only to make variations which are symmetric in the real axis to see this. But since we found in Sections 3, 4, and 5 all the schlicht solutions of (32), and since (104) does not have real coefficients, we deduce that (6), with $n = 3$, is actually the extremal function maximizing b_3 when all the coefficients b_n are real. On the other hand, rotation of (104) through 45° yields the function with real coefficients which minimizes b_3. Thus

for real coefficients we have the peculiar result

$$(109) \qquad -\frac{1}{2} - e^{-6} \le b_3 \le \frac{1}{2} \quad .$$

The exceptionally small difference in the size of the estimates on the left and on the right in (109) is quite remarkable.

After our discussion so far, it would appear that the next problem in order of difficulty is to settle the truth of the conjecture $|a_4| \le 4$ for schlicht functions (1) inside the unit circle. Our success with the inequality (4) would indicate that the most promising approach to this question lies through the study of equations analogous to (85) for the earlier coefficients a_2 and a_3 of the extremal function. We are able here only to describe the nature of these equations.

From the differential equation for the extremal function $w = f(z)$ maximizing $|a_4|$ with $a_4 > 0$, we can derive the identity

$$(110) \qquad \int \left(\frac{1}{w^2} + \frac{3a_2}{w} + 2a_3 + a_2^2\right)^{1/2} \frac{dw}{w^{3/2}}$$

$$= \int \left(\frac{1}{z^3} + \frac{2a_2}{z^2} + \frac{3a_3}{z} + 3a_4 + 3\bar{a}_3 z + 2\bar{a}_2 z^2 + z^3\right)^{1/2} \frac{dz}{z} \quad .$$

With a suitable constant of integration and with the integrals interpreted as indefinite integrals, (110) defines the extremal map $w = f(z)$ implicitly. On the other hand, if we integrate over two corresponding closed paths in the z-plane and in the w-plane, (110) becomes merely a numerical equation. Furthermore, by Cauchy's theorem the closed paths need not correspond according to the map, provided their topology relative to the roots and the poles of the expressions in parentheses is the same. Let w_1 and w_2 be the zeros of the integrand on the left in (110) and let z_1, z_2, $\bar{z}_1^{-1}$, $\bar{z}_1^{-1}$, and $z_3 = \bar{z}_3^{-1}$ be the zeros of the integrand on the right, with $|z_1| < 1$, $|z_2| < 1$.

We obtain two equations, independent of the conformal map $w = f(z)$, for a_2 and a_3 if we choose as the contours of integration in (110) loops around 0 and w_1 and around 0 and z_1, or loops around 0 and w_2 and around 0 and z_2. In order to discuss effectively these two complex equations for a_2 and a_3 it would first be necessary to generalize and refine preliminary estimates on a_2 and a_3 of the type (91).

Finally, we wish to point out that the geometrical analysis of Section 2 would lead rather easily to the result $|a_4| \leq 4$ if it were possible to establish first even such a simple condition of symmetry for the extremal function as $\arg a_3 = \arg a_2^2$.

There is a problem related to coefficient inequalities for the schlicht functions (2) which is concerned with certain diameters D_n associated with a connected bounded closed set Γ. We define

(111) $$D_n = \max\Big[\prod_{i<j} |t_i - t_j|\Big]^{\frac{2}{n(n-1)}}$$

for all choices of the n points t_j lying in the continuum Γ. The number D_2 is the usual diameter of the set Γ, and it can be shown that as $n \to \infty$ the n^{th} diameter D_n approaches the outer mapping radius, or transfinite diameter, e^{δ} of Γ. We are interested in the problem of determining a set Γ which has, for a prescribed value of the outer mapping radius e^{δ}, the largest possible value of D_n [2, 6]. That such an extremal set Γ exists is an easy consequence of the theory of normal families of analytic functions.

It is well known that D_2 is a maximum when Γ is a line segment with end-points t_1 and t_2. We shall give a new proof here that, for $e^{\delta} = 1$, D_3 is a maximum when Γ consists of three equally spaced rays from the origin out to the three cube roots of 4, in which case t_1, t_2, and t_3 lie

at these cube roots.

It takes only a routine application of variational methods to show that the analytic function $p(t)$ whose real part is the Green's function of the exterior of the extremal set Γ maximizing D_3 satisfies the differential equation [6]

$$p'(t)^2 = \frac{t - \frac{t_1+t_2+t_3}{3}}{(t-t_1)(t-t_2)(t-t_3)} \qquad (112)$$

Furthermore, there is no loss of generality if we assume that Γ has been rotated and translated so that

$$t_1 + t_2 + t_3 = 0 \quad , \quad t_1 > 0 \quad . \qquad (113)$$

Thus Γ consists of three analytic arcs which fork from the origin under angles of 120° and terminate at t_1, t_2, and t_3. The problem is to find t_1, t_2, and t_3.

We can find equations for t_1, t_2, and t_3 by noticing that the Green's function vanishes on Γ and hence vanishes at 0, t_1, t_2, and t_3. Thus, in particular,

$$\operatorname{Re} \int_0^{t_1} \frac{t^{1/2}\,dt}{[(t-t_1)(t-t_2)(t-t_3)]^{1/2}} = \operatorname{Re}\left\{ p(t_1) - p(0) \right\} = 0 \quad , \qquad (114)$$

where we are allowed to integrate along the segment $0 < t < t_1$. It follows from (114) that the integrand must be pure imaginary for at least one value t_o of t between 0 and t_1. Hence

$$\operatorname{Im}\left\{ t_o^2 - (t_2 + t_3)t_o + t_2 t_3 \right\} = 0 \quad , \qquad (115)$$

or, since $t_2 + t_3 = -t_1 < 0$,

$$\operatorname{Im}\, t_2 t_3 = 0 \quad . \qquad (116)$$

The relation (116), together with (113), implies that $t_2 = \bar{t}_3$, unless t_2 and t_3 are both real, a case which is easily excluded from the maximum problem by direct calculation. Thus t_1 lies on the perpendicular bisector of the segment joining t_2 and t_3, and since either of the points t_2 or t_3 could as easily have been chosen to be the one lying on the positive real axis, we deduce that t_1, t_2, and t_3 are the vertices of an equilateral triangle whose center lies at the origin.

From the normalization $e^{\gamma} = 1$ we now find that t_1, t_2, and t_3 are, indeed, the cube roots of 4, and (114) shows that Γ consists of three line segments joining these roots to the origin. This proof illustrates how simply conditions of the form (114) can sometimes be treated, even when they involve elliptic integrals.

It can be shown that the four rays from the origin to the fourth roots of 4 do not compose the extremal set maximizing D_4. The proof is too involved to present it profitably here, since the outcome is negative. The construction of a counter-example consists in applying the variation

$$t^* = t + \frac{i\rho^2}{t} \tag{117}$$

to the fourth roots of 4 with a small positive value of ρ and developing in powers of ρ the outer mapping radius and the fourth diameter of the continuum through the varied points which has the smallest possible outer radius. Letting $\rho \to 0$, we find from this development that the varied continuum has a larger value of $D_4 e^{-\gamma}$ than the original symmetric fork had.

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